result of the transformation

$$e_0^2 \rightarrow \epsilon^{-1} e_0^2, \quad \mathcal{G}_{\mu\nu} \rightarrow \epsilon \mathcal{G}_{\mu\nu}, \tag{36}$$

in the limit $\epsilon \rightarrow 0$. The transformation (36) belongs to the renormalization group and leaves the physical content of the theory unchanged.

6. CONCLUSIONS

The main result of this paper, which is that particles may appear in conventional field theory without corresponding fields being introduced into the initial Lagrangian, seems to be quite general. We restricted ourselves to quantum electrodynamics, but very similar results can be obtained in mesodynamics.

There are many formal similarities between our approach and the results of Jouvet,[10] Nambu and Jona-Lasinio,[11] Bjorken,[2] and Freund.[3] It is very likely that all these, seemingly completely different, descriptions are to some extent equivalent.

We do not think that in this paper we have given a proof that the photon is a bound state of an electron-positron pair. We would rather say that our results indicate that an ambiguity exists in the relationship between physical particles and Lagrangians. There may exist several Lagrangians leading to the same set of Feynman diagrams.

[10] B. Jouvet, Nuovo Cimento **5**, 1 (1957).
[11] Y. Nambu and G. Jona-Lasinio, Phys. Rev. **122**, 345 (1961).

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Equilibrium Configuration and Fission Barrier for Liquid Drop Nuclei with High Angular Momentum*

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The equilibrium shape and the fission barrier are calculated for the entire range of angular momenta for which a rotating droplet held together by surface tension has a stable equilibrium. A liquid drop which is originally spherical takes the shape of an oblate spheroid as the angular momentum increases. At higher angular momenta, the shape becomes concave at the poles and a ring form is created. It is shown that the equilibrium ceases to be stable at or near the critical angular momentum at which this change of topology occurs.

I. INTRODUCTION

THIS paper presents a calculation of the equilibrium configurations of rotating liquid drop nuclei and the fission barrier of such drops. The opposing effects of surface tension and centrifugal forces are considered. In this respect the calculation differs from the work of Bohr and Wheeler[1] where nonrotating nuclei were considered and the two opposing effects determining stability were the Coulomb energy and surface tension. The purpose of this paper, in which Coulomb effects are neglected, is to obtain information about the effect of angular momentum on nuclear stability. When the separate effects of angular momentum and Coulomb forces will be known, one might attempt to look at the general case where both effects exist.

The importance of nuclear states with high angular momenta was realized from the results obtained with the heavy-ion accelerators. When uranium is bombarded by 10-MeV oxygen nuclei, states with angular momenta as high as 60 units of $\hbar$ are obtained. It has been found experimentally[2] that the partial width for fission increases with increasing angular momentum.

The properties of a rotating incompressible fluid have been studied and discussed by Plateau,[3] Poincaré,[4] Rayleigh,[5] and Appel.[6] However, none of these papers contain an analytic expression for the equilibrium shape. All of the authors resort to numerical methods at one stage or another. In this paper analytic expressions for the shape of equilibrium are obtained. It is also shown that the topology of the equilibrium configuration changes when a parameter (which will be defined later) assumes the value 2.414. This value is to be compared with 2.32 and 2.4, which are the estimates of Appel and Rayleigh, respectively. The nature of those equilibria with respect to small deformations when the

* Part of this work was submitted as part of a thesis in partial fulfillment for the Ph.D. degree at Princeton University.
[1] N. Bohr and J. A. Wheeler, Phys. Rev. **56**, 426 (1939).

[2] S. A. Baraboshkin, A. S. Karamian, and G. N. Flerov, Soviet Phys.—JETP **5**, 1055 (1957).
[3] J. Plateau, *Mémoire sur les Phénomènes Que Présente Une Masse Liquide Libre et Soustraite de l'Action de la Pesanteur*, Première Partie, Nouveaux Mémoires de l'Academie Royle des Sciences et Belles Lettres de Bruxelles, Tome **16** (1843).
[4] H. Poincaré, *Cappillarité*, George Carré, Editeur (Paris, 1895), pp. 118.
[5] Lord Rayleigh, Phil. Mag. **28**, 161 (1914).
[6] P. Appel, *Traité de Macanique Rationelle* (Gauthier-Villars, Paris, 1932), Vol. 4, Chap. I, p. 295.

angular momentum is kept constant is studied here. According to Lyttleton,[7] Schwarzchild[8] was the first to show that an equilibrium may be unstable when the angular velocity is constant, yet may be stable when the angular momentum remains constant. Hence the two problems have to be investigated separately.

Recently Pik-Pichak,[9] who limited his attention to nuclei which have sphere-like saddle-point configurations and for which the rotational energy is small in comparison with either the surface or Coulomb energy, showed that the fission barrier decreases with the increase of angular momentum. Swiatecki[10] improved on Pik-Pichak's calculation.

A wider problem was investigated by Hiskes,[11] who calculated the forms of equilibrium, although the stability of the equilibrium state and the height of the fission barrier have not yet been calculated. Hiskes used a variation iteration method to generate the equilibrium shapes.

More recently Beringer and Knox[12] calculated the equilibrium shapes of a rotating, uniformly charged liquid drop. In this calculation they limited their attention to spheroidal forms.

In the present paper, exact and explicit formulas are given for the equilibrium forms of rotating chargeless nuclei for the entire range of angular momenta for which stable equilibrium exists, without limitation to spheroidal equilibrium shapes or small rotational energy (Sec. II), the stability of the forms is investigated (Sec. III), and the fission barrier is calculated (Sec. IV).

II. THE THEORY OF EQUILIBRIUM SHAPES AND PROPERTIES OF ROTATING LIQUID DROP NUCLEI AND THEIR PROPERTIES

The dimensionless forms for the energy B_u, square of the angular momentum λ^2, surface area B_s, and inverse of the moment of inertia B_r are defined as

$$B_u = U/(4\pi R_0^2\sigma), \tag{1}$$

$$\lambda^2 = \frac{(I\hbar)^2}{2\times\frac{2}{5}\times(\frac{4}{3}\pi R_0^3)\rho_g R_0^2\times 4\pi R_0^2\sigma}, \tag{2}$$

$$B_s = S/(4\pi R_0^2), \tag{3}$$

$$B_r = \mathcal{J}_s/\mathcal{J}_a. \tag{4}$$

Here U is the energy of the drop, R_0 is the radius of the spherical drop, σ is the surface tension, I is its spin, ρ_g is the nuclear mass density, S is the nuclear surface area, and $\mathcal{J}_s$ and $\mathcal{J}_a$ are the moments of inertia of the spherical nucleus and the actual nucleus. The dimensionless energy can be expressed in terms of the other dimensionless quantities in the following way:

$$B_u = B_s + B_r\lambda^2. \tag{5}$$

A configuration is a shape of equilibrium if there is no difference, to first order, between its energy and the energy of a neighboring configuration. The nucleus is idealized as an incompressible fluid, hence the variational problem is constrained by

$$\Delta V = 0. \tag{6}$$

The equilibrium configurations will be obtained by demanding the vanishing of the variation of $\Delta U - \epsilon\Delta V$, in first order:

$$\Delta U - \epsilon\Delta V = 4\pi R_0^2\sigma\left(\Delta B_u - \frac{\epsilon}{4\pi R_0^2\sigma}\Delta V\right)$$

$$= 4\pi R_0^2\sigma\left(\Delta B_s + \lambda^2\Delta B_r - \frac{\epsilon}{4\pi R_0^2\sigma}\Delta V\right) = 0. \tag{7}$$

First, the differences in B_s, B_r, and V between two neighboring configurations to second order (since the first-order difference is not sufficient for the establishment of the stability of equilibrium) are calculated. Since the shapes of equilibrium are cylindrically symmetric,[4] it will be convenient to use cylindrical coordinates. Darbaux[13] and Wheeler[14] calculated ΔS and ΔV and obtained

$$\Delta S = \int\left[\kappa h + \tfrac{1}{2}(\nabla h)^2 + \frac{h^2}{R_1R_2}\right]dS, \tag{8}$$

$$\Delta V = \int (h + \tfrac{1}{2}\kappa h^2)dS. \tag{9}$$

Here κ is the total curvature, h is the length of a vector perpendicular to a surface of equilibrium measuring the distance from the surface of equilibrium configuration to a close configuration (Fig. 1), ∇h is the gradient of h with respect to a set of orthogonal coordinates on S, dS is an element of surface area, and R_1 and R_2 are the two principal radii of curvature.

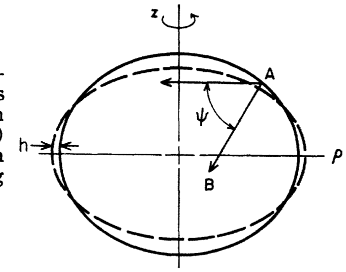

FIG. 1. Schematic diagram of a meridian cross section of an equilibrium configuration (heavy line) and a nearby configuration (broken line) of rotating nucleus.

[7] R. A. Lyttleton, *The Stability of Rotating Liquid Masses* (Cambridge University Press, New York, 1953), p. 26.
[8] N. Schwarzschild, Ann. Munchener Sternwante **3**, 233 (1897).
[9] G. A. Pik-Pichak, Soviet Phys.—JETP **7**, 238 (1958).
[10] W. J. Swiatecki (unpublished).
[11] J. A. Hiskes, University of California Radiation Laboratory Report UCRL-9275, 1960 (unpublished).
[12] R. Beringer and W. J. Knox, Phys. Rev. **121**, 1195 (1961).
[13] G. Darbaux, *Leçons sur la Theorie Générale des Surfaces, et les Applications du Calcul Infinitésimal* (Gauthier-Villars, Paris, 1895), Vol. 1, Chap. III-2.
[14] J. A. Wheeler (private communication).

To calculate ΔB_r to second order in h, $\Delta\mathcal{I}$ has to be calculated to second order in h; one can easily see that

$$\Delta\mathcal{I}=\rho_g\int[\rho^2h+\tfrac{1}{2}(2\rho\cos\psi+\kappa\rho^2)h^2]dS. \tag{10}$$

Using the fact that

$$\Delta B_r=-B_r\left[\left(\frac{\Delta\mathcal{I}}{\mathcal{I}}\right)-\left(\frac{\Delta\mathcal{I}}{\mathcal{I}}\right)^2\right], \tag{11}$$

ΔB_U can be expressed as

$$\Delta B_U=\frac{1}{4\pi R_0^2\sigma}\int\left\{\left[\sigma\kappa-\frac{\rho_g(I\hbar)^2}{2\mathcal{I}^2}\right]h+\frac{\sigma}{2}\left[(\nabla h)^2+\frac{2h^2}{R_1R_2}\right]-\frac{(I\hbar)^2}{4\mathcal{I}^2}[2\rho\cos\psi+\rho^2\kappa]h^2\right\}dS+\frac{(I\hbar)^2}{2\mathcal{I}^3}\left[\int\rho^2hdS\right]^2. \tag{12}$$

To obtain the equilibrium configuration it is necessary to annul the coefficient of h in (7) so that

$$\epsilon=-[(I\hbar)^2/2\partial^2]\frac{1}{\rho_g}+\sigma\kappa$$
$$=-\tfrac{1}{2}\rho_g(\text{velocity})^2+\text{pressure under surface}. \tag{13}$$

Equation (13) may be rewritten in terms of the dimensionless quantities as

$$\kappa R_0-\frac{15}{2}\frac{B_r^2\lambda^2\rho^2}{R_0^2}-\frac{R_0\epsilon}{\sigma}=0. \tag{14}$$

Now ϵ will be expressed in terms of B_s, $B_r\lambda^2$, and R_0 by applying an argument due to Swiatecki.[15] The constant ϵ can be determined by making a uniform change of scale of the equilibrium shape and equating the calculated energy of the new shape based on two calculations: one assuming equal energy density, the other based on dimensional analysis.

$$\Delta E=\int\epsilon dV=\epsilon\bar{\Delta}V, \tag{15}$$

$$\Delta E=E_s\left(\frac{V+\bar{\Delta}V}{V}\right)^{2/3}+E_r\left(\frac{V+\bar{\Delta}V}{V}\right)^{-5/3}-(E_s+E_r). \tag{16}$$

Here $\bar{\Delta}V$ is the volume difference between the two different but similar shapes and E_s and E_r are the surface and rotational energies of the initial configuration of equilibrium. Assuming a small relative change in volume, i.e.,

$$\bar{\Delta}V/V\ll 1, \tag{17}$$

one obtains

$$\epsilon=\frac{\frac{2}{3}E_s-(5/3)E_r}{V}=\frac{(2B_s-5B_r\lambda^2)\sigma}{R_0}. \tag{18}$$

[15] W. J. Swiatecki, Phys. Rev. **104**, 993 (1956).

Consequently, the differential equation (14) may be written as

$$(\kappa R_0-2B_s)-5B_r\lambda^2(1-\tfrac{3}{2}B_r)\rho^2/R_0^2=0, \tag{19}$$

for the case $\lambda=0$, and $B_s=1$, one obtains

$$\kappa=2/R_0 \tag{20}$$

as expected.

Equation (19) may be rewritten in terms of ρ and ψ (Fig. 1) as

$$-\sin\psi\frac{d\psi}{d\rho}+\frac{\cos\psi}{\rho}=\frac{-5B_r\lambda^2}{R_0}\left(1-\tfrac{3}{2}B_r\frac{\rho^2}{R_0^2}\right)+\frac{2B_s}{R_0}. \tag{21}$$

The first integration of (21) yields

$$\cos\psi=\frac{15B_r^2\lambda^2}{R_0^3}\rho^3+\frac{2B_s-5B_r\lambda^2}{R_0}\rho+\frac{C}{\rho}. \tag{22}$$

As long as the shapes of equilibrium are confined to shapes with a topology equivalent to that of a sphere, the constant of integration C in (22) vanishes since

$$\rho=0 \text{ implies } \cos\psi=0. \tag{23}$$

Performing a second integration the following dependence of z on ρ is obtained:

$$z(\rho_1)=\int_{\rho_1}^{R_e}\left\{\left(\frac{15}{8}\frac{B_r^2\lambda^2}{R_0^3}\rho^4+\frac{2B_s-5B_r\lambda^2}{2R_0}\rho^2\right)\Big/\left[\rho-\left(\frac{15}{8}\frac{B_r^2\lambda^2}{R_0^3}\rho^4+\frac{2B_s-5B_r\lambda^2}{2R_0}\rho^2\right)^2\right]^{1/2}\right\}d\rho. \tag{24}$$

In Eq. (24), R_e is the value of ρ at the equator and is determined from volume conservation.

A dimensionless variable x instead of ρ^2 and a dimensionless parameter η measuring the deformation will now be introduced such that

$$\rho^2=R_e^2x, \tag{25}$$

$$\frac{\epsilon}{2\sigma}=\frac{B_s-\frac{5}{2}B_r\lambda^2}{R_0}=\frac{1}{R_e}(1-\eta). \tag{26}$$

Utilizing the fact that

$$(d\rho/dz)_{\rho=R_e}=0, \tag{27}$$

and that for zero angular momentum

$$\epsilon/2\sigma=1/R_0, \tag{28}$$

one obtains

$$\eta=15B_r^2\lambda^2R_e^3/8R_0^3=15B_r^2\lambda^2N^3/8. \tag{29}$$

Using (18), (26), and (29) the following relation between the dimensionless parameters describing the equilibrium configuration is obtained:

$$B_s=(1/N)+\tfrac{5}{2}B_r\lambda^2[1-\tfrac{3}{4}B_rN^2]. \tag{30}$$

This general expression is useful in checking numerical calculations.

The formulas for the function $z(\rho)$, the volume V, the moment of inertia $\mathcal{I}$, and the surface area S of the equilibrium shape can be written as

$$z(\rho)=\tfrac{1}{2}NR_0[\eta I_1(\rho^2/R_e^2)-(\eta-1)I_0(\rho^2/R_e^2)], \quad (31)$$

$$V=\pi N^3R_0^3[\eta I_2-(\eta-1)I_1], \quad (32)$$

$$\mathcal{I}=\tfrac{1}{2}\pi N^5R_0^5\rho_0[\eta I_3-(\eta-1)I_2], \quad (33)$$

$$S=2\pi R_0^2N^2I_0. \quad (34)$$

Here

$$I_n(x_1)=\int_{x_1}^{1}\frac{x^n dx}{[1-(\eta-1)^2x+2(\eta-1)^2x^2-\eta^2x^3]^{1/2}}, \quad (35)$$

$$I_n=I_n(x_1=0). \quad (36)$$

All the integrals I_n for $n\geqslant 2$ may be expressed as linear combinations of I_0, I_1, and a known function of x. Transforming the elliptic integrals I_0 and I_1 into standard forms by means of

$$H=\tfrac{1}{2}+(2/\eta),\quad k^2=\frac{(1/\eta)+\frac{1}{2}+H}{2H}, \quad (37)$$

$$H(k)=\frac{[1/\eta(k)]+\frac{1}{2}}{2k^2-1}, \quad (38)$$

$$\eta=(16k^4-16k^2+2)-2(2k^2-1)(16k^4-16k^2+1)^{1/2}, \quad (39)$$

$$x=[1-H\tan^2(\theta/2)], \quad (40)$$

the parametric dependence between ρ and z becomes

$$\rho(k,\theta)=R_e(k)[1-H\tan^2(\theta/2)]^{1/2}, \quad (41)$$

$$z(k,\theta)=R_e(k)\tfrac{1}{2}[1-HF+2H\bar{E}]. \quad (42)$$

Here $F=F(k,\theta)$ is the elliptic integral of the first kind and

$$\bar{E}=E(k,\theta)-\tan(\theta/2)(1-k^2\sin^2\theta)^{1/2}, \quad (43)$$

where $E(k,\theta)$ is the elliptic integral of the second kind.

The constant $R_e=R_e(k)$ is now determined by demanding volume conservation. For the volume V one obtains

$$V=\frac{\pi R_e^3}{3\eta H^{1/2}}\left[\left(\frac{1}{\eta}-H\right)F+2H\bar{E}+2H^{1/2}\right]$$
$$=\frac{2\pi R_e^3}{3\eta}\left[\frac{z(0)}{R_e}(\eta-1)+1\right]=\tfrac{4}{3}\pi R_0^3, \quad (44)$$

so that

$$N=\frac{R_e}{R_0}=\left[\frac{2\eta}{1+(\eta-1)z(0)/R_e}\right]^{1/3}. \quad (45)$$

In Eqs. (44) and (45), $z(0)$ is the value of z at $\rho=0$.

TABLE I. The relation between ρ and z (coordinates of point on equilibrium shape) for values of deformation parameter η=0.0, 0.5, 1, 1.5, 2.0, 2.414.

$\eta=$ 0.0 ρ	z	0.5 ρ	z	1.0 ρ	z
0.000	1.000	0.000	0.721	0.000	0.545
0.200	0.980	0.308	0.694	0.251	0.545
0.400	0.917	0.605	0.642	0.510	0.535
0.600	0.800	0.821	0.535	0.824	0.484
0.800	0.600	1.041	0.365	1.071	0.362
1.000	0.000	1.120	0.000	1.260	0.000
$\eta=$ 1.5 ρ	**z**	**2.0 ρ**	**z**	**2.414 ρ**	**z**
0.000	0.342	0.000	0.168	0.000	0.000
0.206	0.350	0.306	0.196	0.421	0.065
0.342	0.382	0.688	0.268	0.745	0.171
0.685	0.395	0.995	0.351	1.000	0.290
1.162	0.350	1.302	0.322	1.475	0.381
1.375	0.000	1.531	0.000	1.691	0.000

Similarly B_r and B_s become

$$B_r=\frac{4\eta}{3(\eta-1)N^2+B_sN^3}, \quad (46)$$

$$B_s=\tfrac{1}{2}N^2I_0. \quad (47)$$

The problem of forms of equilibrium with the topology of a sphere is now completely solved in principle.

In Table I and Fig. 2, the results of the calculation of z, ρ dependence in the upper right-hand part of the z, ρ plane (for seven values of η) is given. The values of $H=1$, $\eta=1+\sqrt{2}=2.414$ correspond to $z(0)=0$. For this case $\lambda^2=4$ and the topology of the equilibrium configuration changes. In Table II and Fig. 3, the characteristic properties of the equilibrium configuration for the same values of η are given.

To understand more easily the physical nature of the foregoing complicated formulas, an approximate formula will be obtained valid for those values of the

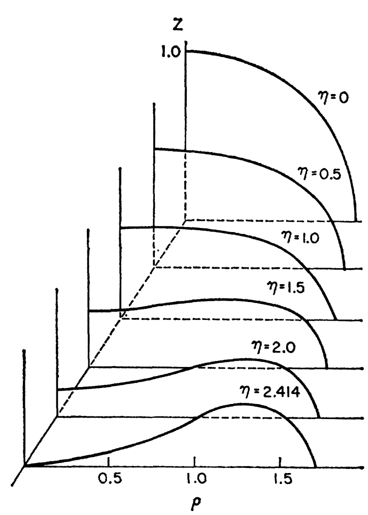

FIG. 2. The upper right-hand corner of the meridian cross section of equilibrium configurations for values of the deformation parameter η =0.0, 0.5, 1.0, 1.5, 2.0, and 2.414.

TABLE II. The characteristic properties of equilibrium configurations as functions of η.

η	N	$z(0)/R_e$	B_r	B_s	λ^2	$B_r\lambda^2$	B_u	$1+\lambda^2$
0.0	1.000	1.000	1.000	1.000	0.000	0.000	1.000	1.000
0.5	1.120	0.645	0.760	1.021	0.315	0.239	1.270	1.315
1.0	1.260	0.432	0.596	1.114	0.750	0.447	1.627	1.750
1.5	1.375	0.251	0.483	1.222	1.468	0.713	1.963	2.468
2.0	1.531	0.112	0.344	1.489	2.650	0.912	2.401	3.650
2.41	1.691	0.000	0.257	1.764	4.000	1.028	2.792	5.000

angular momentum for which the rotational energy is small in comparison to the surface energy. The approximate formulas for volume and the moment of inertia will be obtained by expanding the values previously obtained into a power series in η around $\eta=0$. This procedure yields

$$V=\frac{4\pi}{3}R_0^3N^3\left(1-\eta+\frac{6}{5}\eta^2\right)=\frac{4\pi}{3}R_0^3, \tag{48}$$

$$\mathcal{I}=\frac{8\pi}{15}R_0^5N^5\rho_0\left(1-\eta+\frac{9}{7}\eta^2\right). \tag{49}$$

Eliminating N and using (29), (30), and (49), B_r, B_s, and B_u become

$$B_r=1-\tfrac{5}{8}\lambda^4, \tag{50}$$

$$B_s=1+(5/4)\lambda^4, \tag{51}$$

$$B_u=1+\lambda^2-\tfrac{5}{8}\lambda^4. \tag{52}$$

Replacing the dimensionless quantities by physical quantities, Eq. (52) becomes

$$U=4\pi R_0^2\sigma+\frac{(I\hbar)^2}{2\mathcal{I}_0}\left[1-\frac{5}{8}\frac{(I\hbar)^2}{2\mathcal{I}_0 4\pi R_0^2\sigma}\right],$$

$$=E_s^0+E_r^0\left[1-\frac{5}{8}\frac{E_r^0}{E_s^0}\right]. \tag{53}$$

Here E_s^0 and E_r^0 are the surface and rotational energy of a spherical nucleus. As expected, the energy values in (53) are smaller than those for a spherical nucleus with the same angular momentum. Equation (53) also proves that the higher the surface tension, the more difficult it is to deform the nucleus, and the closer will its energy be to that of a rotating spherical nucleus. Of main interest to this study is that part of the energy which increases with angular momentum and which is

$$U_R=U-E_s^0=E_r^0\left(1-\frac{5}{8}\frac{E_r^0}{E_s^0}\right). \tag{54}$$

The results of applying (54) to some typical nuclei are given in Table III. It has to be stressed again that the Coulomb energy is neglected in deducing (54).

Angular momenta as high as those listed in the last entries of Table II can hardly be reached since neutron leakage is energetically possible for an angular momentum much lower than the critical angular momentum at which the topology of the equilibrium changes. Neutron leakage is feasible for values of about $I=20$ for medium nuclei and about $I=40$ for heavy nuclei (see Table III). While the change of topology demands an I of 500 to 800.

TABLE III. Energy of lowest state with a specified angular momentum as a function of that angular momentum (energies are in MeV, angular momentum in units of $\hbar$). The following constants were used: $r_0=1.216\times10^{-13}$ cm, $4\pi R_0^2\sigma=17.80$ MeV. The calculation is based on the approximate formula (53).

$A=100$		$A=150$		$A=200$		$A=250$	
I	U_r	I	U_r	I	U_r	I	U_r
0	0.000	0	0.000	0	0.000	0	0.000
4	0.258	5	0.209	7	0.253	8	0.228
8	1.030	10	0.833	14	1.010	16	0.910
12	2.313	15	1.874	21	2.271	24	2.047
16	4.099	20	3.326	28	4.021	32	3.636
20	6.376	25	5.184	35	6.280	40	5.698

TABLE IV. The function b_u for $\lambda^2=0.000$, 0.315, 0.750, 1.468, 2.650, and 4.000.

λ^2	0.000 $b_\eta(u)$	0.315 $b_\eta(u)$	0.750 $b_\eta(u)$	1.468 $b_\eta(u)$	2.650 $b_\eta(u)$	4.000 $b_\eta(u)$
0.0	1.000	1.315	1.750	2.468	3.650	5.000
0.5	1.021	1.270	1.657	2.137	3.035	4.061
1.0	1.114	1.285	1.627	1.989	2.693	3.498
1.5	1.222	1.381	1.661	1.963	2.468	2.980
2.0	1.489	1.612	1.738	1.973	2.401	2.830
2.414	1.764	1.849	1.956	2.132	2.436	2.792

III. THE STABILITY OF THE EQUILIBRIUM CONFIGURATION

The shapes just calculated are configurations of equilibrium but the question arises as to whether they are configurations of stable equilibrium. An equilibrium configuration will possess relative stability if its energy for a fixed angular momentum is smaller (to order higher than the first) than any neighboring configuration. Now the appropriate neighboring configurations have to be chosen. These configurations can be chosen in an infinite number of ways, the only limitation being the volume conservation. The configurations corresponding to equilibrium configurations of close angular momenta suggest one possibility. Therefore, the energy b_u of a droplet of a fixed angular momentum parameter λ^2 is calculated as a function of η. In dimensionless form this energy b_u has the value

$$b_u(\eta)=B_s(\eta)+\lambda^2B_r(\eta). \tag{55}$$

Obviously, b_u is at minimum or maximum for the value of η that characterizes the appropriate equilibrium configuration. Table IV and Fig. 4 present the dimensionless energy b_u as a function of the deformation parameter η for various values of λ^2.

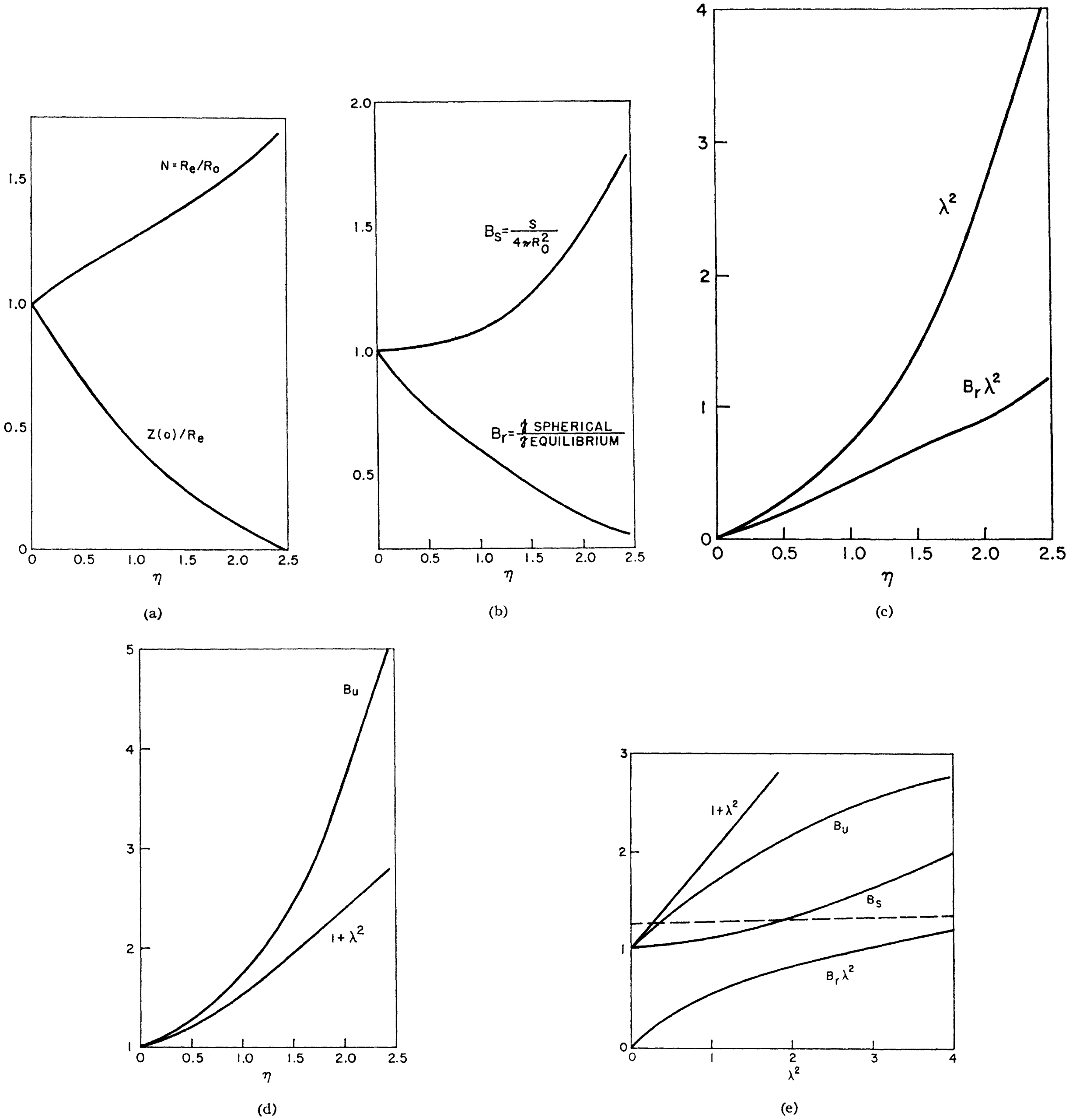

FIG. 3(a). The normalization constant R_e/R_0 and the value of $z(0)/R_e$ as a function of the deformation parameter η. (b). The ratio between the moment of inertia of a spherical configuration and an equilibrium configuration $B_r = \mathcal{I}_{\text{spherical}}/\mathcal{I}_{\text{equilibrium}}$ and the ratio between the surface area of the equilibrium configuration to the area of a sphere $B_s = S/(4\pi R_0^2)$ as a function of the deformation parameter. (c). The square of the dimensionless angular momentum $\lambda^2 = (I\hbar)^2/[2\times\frac{2}{5}\times(4\pi R_0^5/3)\rho_0 4\pi R_0^2\sigma]$ and the dimensionless kinetic energy $B_r\lambda^2 = (I\hbar)^2/(2\mathcal{I}4\pi R_0^2\sigma)$ as a function of the deformation parameter. (d). The dimensionless energy of a configuration of equilibrium $B_u = U/(4\pi R_0^2\sigma)$ and the dimensionless energy of a sphere of the same volume and angular momentum $1+\lambda^2$ as a function of the deformation parameter η. (e). The dimensionless surface energy $B_s = S/(4\pi R_0^2)$, the dimensionless kinetic energy $B_r\lambda^2 = (I\hbar)^2/[2\mathcal{I}(4\pi R_0^2\sigma)]$, and the dimensionless total energy $B_u = U/(4\pi R_0^2\sigma)$ as a function of the dimensionless square of the angular momentum $\lambda^2 = (I\hbar)^2/[2\times\frac{2}{5}(4\pi R_0^5/3)\rho_0(4\pi R_0^2\sigma)]$. The dashed line indicates the dimensionless energy of two droplets of half the mass at infinite separation.

This presentation (see Fig. 4) suggests that with respect to the particular kind of neighboring configurations the equilibrium is stable and that at $\lambda^2 = 4$ ($\eta = 1+\sqrt{2}$) a change of stability might occur.

To establish stability, in principle one has to pick out of the class of volume preserving neighboring configurations the one which has the lowest potential energy, to second order, and convince oneself that its

TABLE V. $1000\Lambda/N^2$ as a function of the deformation parameters η for $\nu=1, 2, 3$ using trial functions with 5 and 7 parameters. The parameter Λ/N^2 is a measure of the stability of an equilibrium form of deformation parameter (small η, slow rotation; large η, fast rotation) against deformation including a term proportional to $\cos\nu\phi$.

$\nu=$	1		2		3	
No. of parameters / η	5	7	5	7	5	7
			$1000\Lambda/N^2$			
0.0	102.0	92.1	72.1	65.2	152.3	130.1
0.5	75	67.2	58.5	54.5	104.5	92
1.0	47.8	47.9	32.3	26.7	73.1	65.3
1.5	35.5	21.3	18.5	14.1	49.2	43.7
2.0	20.6	16.5	8.9	5.8	39.4	24.5
2.41	8.2	5.1	5.0	0.6	13.1	10.2

energy exceeds the energy of the equilibrium configuration. The demand that the energy is smallest to second order specifies the shape uniquely, by demanding

$$\delta\left(\frac{\Delta U}{K}\right)=\left(\int U(h)dS\Big/\int h^2dS\right)=0. \qquad (56)$$

As in many quantum-mechanical problems an exact solution of (56) which would determine h (see Fig. 1) is extremely difficult. Therefore, the Rayleigh-Ritz[16,17] approximation will be used. Using this method a special form for h is chosen which depends on one or more free parameters. Next, the value of the parameters will be established by demanding that the derivative of $\Delta U/K$ with respect to each of these parameters shall vanish. Finally the value of $\Delta U/K$ is calculated for the parameters just found as a function of η. When $\Delta U/K$ becomes negative, the equilibrium is unstable since

$$(\Delta U/K)_{\text{calc}} \gtrless (\Delta U/K)_{\text{true min}}. \qquad (57)$$

It must be remembered that it is still necessary to satisfy the subsidiary condition of volume conservation so that the quantity R to be extremized is

$$R=(\Delta U-\epsilon_v\Delta V)/K. \qquad (58)$$

Here ϵ_v is a Lagrange multiplier. If R is split into first and second terms

$$R=\frac{\Delta U-\epsilon\Delta V_1}{K}+\frac{\Delta U_2-\epsilon\Delta V_2}{K}-\frac{\bar{\epsilon}\Delta V_1}{K}, \qquad (59)$$

where

$$\bar{\epsilon}=\epsilon_v-\epsilon, \qquad (60)$$

and ϵ is defined by (13). Therefore, the first term on the right-hand side of (59) vanishes.

Using (9) and (12), one obtains

$$\Delta U_2-\epsilon V_2=\int\left[\tfrac{1}{2}\sigma(\nabla h)^2+\frac{\sigma h^2}{R_1R_2}-\frac{\rho_g(I\hbar)^2}{4\mathcal{I}^2}(2\rho\cos\psi+\rho^2\kappa)^2\right]dS+\frac{(I\hbar)^2}{2\mathcal{I}^3}\left[\int\rho^2hdS\right]^2. \qquad (61)$$

Here the integration is carried out over a surface of an equilibrium shape. The function h is chosen as

$$h=R_e\textstyle\sum_\nu f_\nu(\rho^2)\cos\nu\phi. \qquad (62)$$

This trial function satisfies the boundary conditions. Furthermore, the following form for the $f_\nu(\rho^2)$ is assumed {expressed as a function of the dimensionless parameter x [defined in Eq. (25)]}:

$$f_0=f_0^0+f_0^1x+f_0^2x^2+f_0^3x^3, \qquad (63)$$

$$f_{\nu\neq0}=f_\nu^1x+f_\nu^2x^2+f_\nu^3x^3. \qquad (64)$$

Here the $f_\mu{}^i$ are the free-disposable parameters, to be determined by the Rayleigh-Ritz approximation. Expressing $(\Delta U_2-\epsilon\Delta V_2)$, ΔV_1, ΔV_2, and K as a function of the $f_\mu{}^i$. One obtains

$$\Delta U_2-\epsilon\Delta V_2=\sigma R_0^2N^2\sum_{ij\mu\nu}U_{ij}{}^{\mu\nu}f_\mu{}^if_\nu{}^j=\sigma R_0^2N^2U_2, \qquad (65)$$

$$\Delta V_1=R_0^3N^3\textstyle\sum_i V_if_0{}^i=R_0^3N^3V_1, \qquad (66)$$

$$\Delta V_2=R_0^3N^3\sum_{ij\mu\nu}V_{ij}{}^{\mu\nu}f_\mu{}^if_\nu{}^j=R_0^3N^3V_2, \qquad (67)$$

$$K=R_0^4N^4\sum_{ij\mu\nu}K_{ij}{}^{\mu\nu}f_\mu{}^if_\nu{}^j=N^4R_0^4K. \qquad (68)$$

In the previous expressions the coefficients of the $f_\mu{}^i$ are known functions of η. The quantity R to be ex-

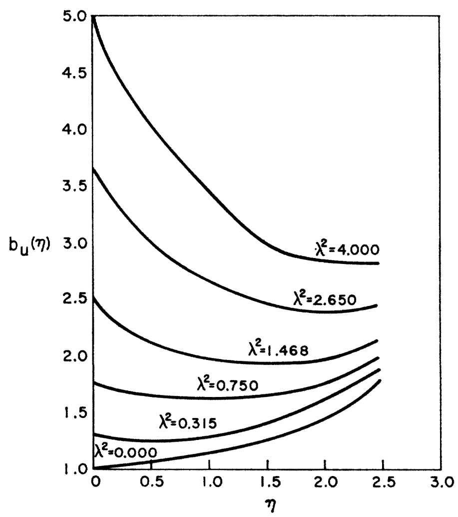

FIG. 4. The dimensionless energy $b_u(\eta)$ as a function of the deformation parameter η for selected values of the square of the angular momentum λ^2. The deformation parameter η picks out one or another form which would be an equilibrium configuration if it were turning with the right angular momentum.

[16] W. Ritz, Ann. Physik 28, 737 (1909).
[17] Lord Rayleigh, Proc. London Math. Soc. 4, 357 (1874).

tremized becomes

$$R=\frac{\sigma}{R_0^2N^2}\left(\frac{U_2}{K}-\alpha\frac{V_1}{K}\right)=\frac{\sigma}{R_0^2N^2}\bar{R}(\alpha,f_\mu{}^i,\eta). \quad (69)$$

Here α is a new Lagrange multiplier. To minimize R one must demand

$$\frac{\partial R}{\partial f_\mu{}^k}=\frac{\sigma}{R_0^2N^2}\left(\frac{1}{K}\frac{\partial U_2}{\partial f_\mu{}^k}-\frac{U_2}{K^2}\frac{\partial K}{\partial f_\mu{}^k}-\frac{\alpha}{K}\frac{\partial V_1}{\partial f_\mu{}^k}+\frac{\alpha V_1}{K^2}\frac{\partial K}{\partial f_\mu{}^k}\right)=0. \quad (70)$$

Since $K\neq 0$ [from Eq. (56)], then Eq. (70) may be written as

$$\frac{\partial U_2}{\partial f_\mu{}^k}-\Lambda_1\frac{\partial K}{\partial f_\mu{}^k}-\Lambda_2\frac{\partial V_1}{\partial f_\mu{}^k}=0. \quad (71)$$

Here

$$\Lambda_1=(U_1-\alpha V_1)/K, \quad \Lambda_2=\alpha. \quad (72)$$

Finally,

$$\sum_{i\mu} U_{ij}{}^{\mu\nu}f_\mu{}^i-\Lambda_1K_{ij}{}^{\mu\nu}f_\mu{}^i-\Lambda_2\delta_{\mu,0}V_j=0. \quad (73)$$

Now new $f_0{}^i$ are defined:

$$f_{0,\text{old}}{}^i=f_{0,\text{new}}{}^i+\beta^i, \quad (74)$$

the β_i satisfying

$$\sum_i[U_{ij}{}^{00}-\Lambda_1K_{ij}{}^0]\beta^i-\Lambda_2V_j=0. \quad (75)$$

Equation (75) determines the β^i in terms of the U, K, V, Λ_1. By introducing the new $f_0{}^i$, the equation for the $f_\mu{}^i$ becomes

$$\sum_{i\mu}[U_{ij}{}^{\mu\nu}-\Lambda_1K_{ij}{}^{\mu\nu}]f_\mu{}^i=0. \quad (76)$$

To obtain a nontrivial solution of (76) one must demand that the determinant of the coefficients of the $f_\mu{}^i$ vanishes. In this way a secular equation is obtained.

As a result of the orthogonality of the trigonometric functions the U's and K's are diagonal in their superscripts such that

$$V_{ij}{}^{\mu\nu}=\delta_{\mu\nu}U_{ij}{}^{\mu\nu}, \quad K_{ij}{}^{\mu\nu}=\delta_{\mu\nu}K_{ij}{}^{\mu\nu}. \quad (77)$$

The diagonality breaks the secular determinant into products of determinants. The advantage of introducing the new eigenvectors (74) is a result of their dependence on Λ_1 only, which is obtained by solving the secular equation (76); Λ_2 may be obtained by demanding the vanishing of $\Delta V/K$ to second order. This yields

$$\Lambda_2=\Lambda_2(\Lambda_1). \quad (78)$$

Finally one obtains for

$$\frac{\Delta U}{K}=\frac{\sigma}{N^2R_0^2}[\Lambda_1+\Lambda_2\Delta V_1]=\frac{\sigma}{R_0^2}\Lambda. \quad (79)$$

Note that ΔV_1 is not zero but instead only the sum $\Delta V_1+\Delta V_2$ vanishes. To obtain the lowest eigenvalue the pair of Λ_1 and Λ_2 is picked which yield the smallest value for Λ. The problem of testing stability, therefore, reduces to testing whether the smallest Λ is positive.

The number of parameters to be adjusted might seem to be either three or four [(63), (64)] depending on whether one deals with a deformation of $\nu\neq 0$ or one of the order $\nu=0$. This is not the case, however, since to describe a deformation of order $\nu\neq 0$ one uses three parameters for that part of the deformation which is proportional to $\cos\nu\phi$ but preservation of the volume requires also an accompanying deformation independent of ϕ and for this four parameters are used. The variational calculation was performed for $\nu=1$, 2, 3 and trial functions with a total of five and seven parameters were used. The results of this variational calculation are given in Table IV.

As expected, the trial wave functions with seven parameters yield smaller eigenvalues than those with five parameters. The function with $\nu=2$ yields the smallest eigenvalues. It appears reasonable to expect that, with better trial functions including more disposable parameters, Λ would vanish for $\eta=1+\sqrt{2}$. Therefore, it can be concluded that, for $\eta=1+\sqrt{2}$, when the topology changes from sphere-like to ring-like, the equilibrium changes from stable to unstable. Rayleigh[17] suspected that this would be the case. The conclusion that the lowest energy is obtained when the deformation is a combination of ϕ-independent term and a term depending on $\cos 2\phi$ will be utilized in calculating fission barriers.

IV. FISSION BARRIER AS A FUNCTION OF ANGULAR MOMENTUM FOR A ROTATING NUCLEUS

The method of Bohr and Wheeler[1] in their treatment of fission of nonrotating nuclei will be adapted to rotating nuclei. The fission barrier is calculated for (a) nuclei with very small angular momenta, and (b) for nuclei with angular momenta close to $\lambda^2=4$ or $\eta=1+\sqrt{2}$, at which the nature of the stability of the equilibrium configuration changes from stable to unstable. By means of interpolation for intermediate values, the fission barrier is obtained for the entire range of angular momenta for which the equilibria configurations have a sphere like topology. Starting with zero angular momentum the saddle-point configuration consists of two spheres each of equal mass and radii R_1. The fission barrier E_f is the necessary energy to break the original spherical nucleus into two spheres, so that

$$E_f=2\times 4\pi R_1^2\sigma-4\pi R_0^2\sigma=4\pi R_0^2(2^{1/3}-1). \quad (80)$$

If there is a small nonvanishing angular momentum, the configuration leading to fission (saddle-point configuration) will consist of two spheres connected by a thin neck of nuclear fluid. The ratio r_n of the radius of the neck to the radius of the sphere is shown to be of the first order in λ^2. Consequently, it will have only second-order effects on the fission barrier which are neglected in the present calculations. To estimate the order of magnitude of r_n, one has to compare the attractive force due to surface tension trying to keep the spheres together with the centrifugal force tearing them apart. Therefore,

$$\frac{r_n}{R_0}=\left(\frac{\mathscr{I}_0}{\mathscr{I}_1}\right)\frac{5}{2\mathscr{I}_0}\frac{(I\hbar)^2}{2R_0^2 2\pi\sigma 2^{1/3}}=\frac{10}{49}\frac{2^{10/3}}{2^{1/3}}\lambda^2=\frac{80}{49}\lambda^2, \quad (81)$$

where $\mathscr{I}_1$ is the moment of inertia of the saddle-point configuration. Now the fission barrier to first order in λ^2 becomes

$$E_f=E_r^s+E_s^s-E_s-E_r=\frac{2\times 4\pi R_0^2\sigma}{2^{1/3}}+\frac{4\pi R_0^2\sigma 2^{5/7}}{7}\lambda^2 -4\pi R_0^2\sigma-4\pi R_0^2\sigma\lambda^2=4\pi R_0^2\sigma(0.261-0.5463\lambda^2). \quad (82)$$

Here E_r^s and E_s^s are the rotational and surface energy, respectively, of the saddle-point configuration.

Next the fission barrier for angular momentum close to the angular momentum at which the nature of equilibrium changes is calculated. Since the saddle-point configuration does not differ appreciably from the equilibrium, configuration parameters (small with respect to 1) will be used to measure departure from equilibrium. Thus, it will suffice to calculate the energy to the fourth power in the deformation parameters. The deformed shapes are determined by

$$\rho(z,\varphi)=\rho_0(z)[1-\gamma_0+\gamma_1\cos 2\theta+(\gamma_2\rho_0^2/R_e^2)\cos 2\phi], \quad (83)$$

and are now investigated. Here $\rho_0(z)$ describes the equilibrium configuration [$\rho_0(z)$ is $\rho(z)$ of Sec. 2]. The three gammas are not independent due to volume conservation. To begin with, the volume V_d of the deformed configuration is calculated as

$$V_d=\int\int\rho_0(z)^2\left[1-\gamma_0+\gamma_1\cos 2\phi+\frac{\gamma_2\rho_0^2}{R_e^2}\cos 2\phi\right]^2 d\phi dz$$
$$=V\left[1-2\gamma_0+\gamma_0^2+\frac{\gamma_1^2}{2}+\frac{I_3-(\eta-1)I_2}{I_2-(\eta-1)I_1}+\gamma_2^2\frac{I_4-(\eta-1)I_3}{I_2-(\eta-1)I_1}\right]=V. \quad (84)$$

Here V is the volume of the equilibrium configuration and the integrals I_n have been defined in (35). Demanding volume conservation, i.e., $V_d=V$, and solving for γ_0, one obtains

$$\gamma_0=v_{11}\gamma_1^2+v_{12}\gamma_1\gamma_2+v_{22}\gamma_2^2+V_{1111}\gamma_1^4+V_{1112}\gamma_1^3\gamma_2 +V_{1122}\gamma_1^2\gamma_2^2+V_{1222}\gamma_1\gamma_2^3+V_{2222}\gamma_2^4. \quad (85)$$

The linear and cubic terms disappear as a result of the integration over the azimuthal angle ϕ.

All the v's and V's are functions of η only. The moment of inertia can be calculated in a similar way. After replacing γ_0 and γ_0^2 from (85) one gets

$$I_d=i_{11}\gamma_1^2+i_{12}\gamma_1\gamma_2+i_{22}\gamma_2^2+I_{1111}\gamma_1^4+I_{1112}\gamma_1^3\gamma_2 +I_{1122}\gamma_1^2\gamma_2^2+I_{1222}\gamma_1\gamma_2^3+I_{2222}\gamma_2^4. \quad (86)$$

Here again the coefficients of the γ are functions of η only. The calculation of the surface area S_d of the deformed form deserves a little more attention.

$$S_d=\int\int\rho(z,\varphi)\left[1+\left(\frac{\partial\rho}{\partial z}\right)^2+\left(\frac{1}{\rho}\frac{\partial\rho}{\partial\varphi}\right)^2\right]^{1/2} d\phi dz. \quad (87)$$

Defining

$$\rho^2(z,\phi)=\rho_0^2[1+F_1(\phi,\gamma,\rho_0^2)], \quad (88)$$

$$\left(\frac{\partial\rho}{\partial z}\right)^2=\left(\frac{\partial\rho_0}{\partial z}\right)^2[1+F_2(\phi,\gamma,\rho_0^2)], \quad (89)$$

$$\left(\frac{1}{\rho}\frac{\partial\rho}{\partial z}\right)^2=F_3(\phi,\gamma,\rho_0^2). \quad (90)$$

Here F_1, F_2, and F_3 are known polynomials in ρ_0^2 and since they include only terms with γ, the F's, too, are small with respect to 1. Using (88), (89), and (90) and expanding (87) into a power series in the gammas, S_d becomes

$$S_d=\int\int\rho_0^2\left[1+\left(\frac{\partial\rho_0}{\partial z}\right)^2\right]^{1/2}[1+F_1]$$
$$\times\left\{1+\frac{1}{2}\frac{1}{1+(\partial\rho_0/\partial z)^2}\left[\left(\frac{\partial\rho_0}{\partial z}\right)^2F_2+F_3\right] -\frac{1}{8}\frac{1}{[1+(\partial\rho_0/\partial z)^2]^2}\left[\left(\frac{\partial\rho_0}{\partial z}\right)^2F_2+F_3\right]^2\right.$$
$$+\frac{1}{16}\frac{1}{[1+(\partial\rho_0/\partial z)^2]^3}\left[\left(\frac{\partial\rho_0}{\partial z}\right)^2F_2+F_3\right]^3 \left.-\frac{5}{128}\frac{1}{[1+(\partial\rho_0/\partial z)^2]^4}\left[\left(\frac{\partial\rho_0}{\partial z}\right)^2F_2+F_3\right]^4\right\}d\phi dz. \quad (91)$$

It was already mentioned that the F's are polynomials in ρ_0^2. It can be easily shown that $(\partial\rho_0/\partial z)^2[1+(\partial\rho_0/\partial z)^2]^{-1}$ and $[1+(\partial\rho_0/\partial z)^2]^{-1}$ are also polynomials in ρ_0^2 (or in x). After eliminating γ_0 and γ_0^2, S_d becomes

$$S_d=s_{11}\gamma_1^2+s_{12}\gamma_1\gamma_2+s_{22}\gamma_2^2+S_{1111}\gamma_1^4+S_{1112}\gamma_1^3\gamma_2 +S_{1122}\gamma_1^2\gamma_2^2+S_{1222}\gamma_1\gamma_2^3+S_{2222}\gamma_2^2. \quad (92)$$

The difference in energy between the deformed shape and the initial equilibrium configuration ΔB_u will, thus, be

$$\Delta B_u=B_s\frac{S_d-S_e}{S_e}+B_r\lambda^2\left[\frac{1}{\mathscr{I}_d}-\frac{1}{\mathscr{I}_e}\right]. \quad (93)$$

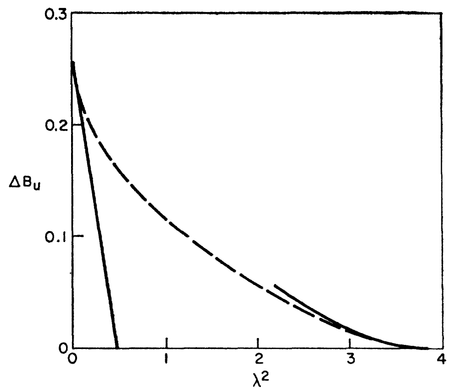

FIG. 5. The fission barrier as a function of angular momentum. Heavy line on left as calculated for small angular momenta; curve on right as calculated for λ^2 near 4. The dashed curve in between is obtained by interpolation.

The above expression (93) will be calculated up to fourth power in γ_1 and γ_2. The coefficients of the γ's are functions of B_s, $B_r\lambda^2$, and η. However, these quantities are not independent and B_s, B_r, and η can be considered as a function of λ^2. Therefore, (93) becomes

$$\Delta B_u = b_{11}\gamma_1^2 + b_{12}\gamma_1\gamma_2 + b_{22}\gamma_2^2 + B_{1111}\gamma_1^4 + B_{1112}\gamma_1^3\gamma_2^2 + B_{1122}\gamma_1^2\gamma_2^2 + B_{1122}\gamma_1\gamma_2^3 + B_{2222}\gamma_2^4, \quad (94)$$

where the b's and the B's are functions of λ^2 only.

The necessary conditions for the maximum ΔB_u are

$$\frac{\partial(\Delta B_u)}{\partial\gamma_1} = \frac{\partial(\Delta B_u)}{\partial\gamma_2} = 0. \quad (95)$$

Introducing the values of γ_1, γ_2, obtained by solving (95), the maximum value of ΔB_u is a function of λ^2 only.

$$(\Delta B_u)_{\max} = \Delta B[b(\lambda^2); B(\lambda^2)], \quad (96)$$

$$E_f = 4\pi R_0^2 \sigma (\Delta B_u)_{\max}. \quad (97)$$

Expanding E_f near $\lambda^2 = 4$ and calculating the coefficients of $(\lambda^2 - 4)^n$ numerically, one obtains

$$(\Delta B_u)_{\max} = 0.0003(\lambda^2 - 4) - 0.0005(\lambda^2 - 4)^2 + 0.1854(\lambda^2 - 4)^3 - 0.0623(\lambda^2 - 4)^4. \quad (98)$$

There is reason to believe that the first two terms are zero, rather than nonvanishing (the error is attributed the numerical calculation) are of the order of magnitude of these terms. This view is supported by the fact that in Bohr-Wheeler's[1] treatment of the fission barrier of nonrotating nuclei the power series starts with a cubic in $(1-x)$. It also follows from a more general argument introduced by Swiatecki[18] that the power series should start with $(1-\lambda^2)^3$. In Fig. 5, the fission barrier is plotted as a function of the angular momentum λ^2. The solid straight line represents the fission barrier near $\lambda^2 = 0$; the solid curve represents the fission barrier for λ^2 close to 4; the dashed line in between is the interpolated fission barrier. In view of the recent work by Swiatecki and Cohen[19] the justification for the interpolation should be investigated.

V. CONCLUSIONS

The originally spherical equilibrium form assumes the shape of an oblate spheroid with an increase in angular momentum. As the angular momentum is increased further the equilibrium shape becomes concave at the poles and finally reaches a stage at which the two poles coalesce and form a ring.

When the topology of the equilibrium shape changes, the equilibrium becomes unstable.

ACKNOWLEDGMENT

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APPENDIX

It should be noted that the result obtained in this paper for the critical angular momentum $\lambda^2 = 4$ at which the nature of stability changes is not in accord with the result obtained by Carlson and Pau Lu.[20] According to their finding, oblate spheroids lose stability at much smaller values of the angular momentum. However, it should be mentioned that whereas Carlson and Pau Lu consider ellipsoidal forms, the forms of equilibrium studied in the present paper are not such. Furthermore, for higher angular momenta they are not even spheroids. One might argue, therefore, that for the same values of the angular momentum for which an oblate cylindrical symmetric ellipsoidal is unstable with respect to small deformation, a different cylindrical symmetric shape may be stable.

[18] W. J. Swiatecki, Phys. Rev. **101**, 657 (1956).
[19] S. Cohen and W. J. Swiatecki (unpublished).
[20] B. C. Carlson and Pau Lu, in *Proceedings of the Rutherford Jubilee International Conference, Manchester*, 1961, edited by J. B. Berks (Heywood and Company, Ltd., London, 1961), p. 291.